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# One Criterion on a Class of Certain Analytic Functions

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Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{z ; |z| < 1\}$.

A function belonging to $A$ is said to be a member of the class $S(\alpha)$ if it satisfies

$$\frac{zf'(z)}{f(z)} \qquad 1 + (1-\alpha)z \tag{1}$$

for some $\alpha\ (0 \leqq \alpha < 1)$ and for all $z \in U$. The symbol denotes the subordination. It is easily confirmed that the condition (1) is equivalent to the following

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \tag{2}$$

for all $z \in U$.

In [1], Fukui obtained the following result

**Theorem A.** If $f(z) \in A$ satisfies

$$\left| \beta \frac{zf'(z)}{f(z)} - 1 + (1-\beta) \frac{zf''(z)}{f(z)} \right| < 1 - \alpha \tag{3}$$

for some $\alpha\ (0 \leqq \alpha < 1)$, $\beta\ (0 \leqq \beta < 1)$, and for all $z \in U$, then $f(z) \in S(\alpha)$.

Making a lemma, we will improve Theorem A.

In order to derive our result, we need the following lemma due to Jack[2] (or Miller and Mocanu[3]).

**Lemma 1.** Let $w(z)$ be analytic in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0$, then we have

$$z_0 w'(z_0) = k w(z_0)$$

where $k$ is real and $k \geqq 1$.

Applying Lemma 1, we have

**Main Theorem.** Let $p(z)$ be analytic in $U$, $p(0) = 1$ and suppose that

$$|\beta(p(z) - 1) + (1-\beta)(p^2(z) - p(z) + zp'(z))| < (1-\alpha)(1+\alpha-\alpha\beta) \tag{4}$$

for some $\alpha\ (0 \leqq \alpha < 1)$, $\beta\ (0 \leqq \beta < 1)$ and for all $z \in U$. Then we have

$$|p(z)-1|<1-\alpha$$

for all $z \in U$.

Proof. Let us put

$$(1-\alpha)w(z)=(p(z)-1).$$

Then we have $w(0)=0$.

By an easy calculation, we have

$$|\beta(p(z)-1)+(1-\beta)(p^2(z)-p(z)+zp'(z))|$$
$$=|\beta(1-\alpha)w(z)+(1-\beta)(1-\alpha)\ \{(1-\alpha)w^2(z)+w(z)+zw'(z)\}\ |$$
$$=\left|(1-\alpha)w(z)\left\{1+(1-\alpha)(1-\beta)w(z)+(1-\beta)\frac{zw'(z)}{w(z)}\right\}\right|$$

If there exists a point $z_0$ such that

$$\max_{z<z_0}|w(z)|=|w(z_0)|=1,$$

then from Lemma 1, we have

$$\left|(1-\alpha)w(z_0)\left\{1+(1-\beta)((1-\alpha)w(z_0)+\frac{z_0w'(z_0)}{w(z_0)})\right\}\right|$$
$$=(1-\alpha)|1+k(1-\beta)+(1-\alpha)(1-\beta)w(z_0)|$$
$$\geqq(1-\alpha)(1+1-\beta-(1-\alpha)(1-\beta))$$
$$=(1-\alpha)(1+\alpha-\alpha\beta).$$

This contradicts to (4). This shows that

$$|p(z)-1|<1-\alpha$$

for all $z \in U$. This completes our proof.

Putting

$$p(z)=\frac{zf'(z)}{f(z)}$$

then we have

$$p^2(z)-p(z)+zp'(z)=\frac{zf''(z)}{f(z)}.$$

Therefore, from the Main theorem, we have

C o r o l l a r y 1. If $f(z) \in A$ satisfies

$$\left|\beta\frac{zf'(z)}{f(z)}-1+(1-\beta)\frac{z^2f''(z)}{f(z)}\right|<(1-\alpha)(1+\alpha-\alpha\beta)$$

for some $\alpha\,(0\leqq\alpha<1)$, $\beta\,(0\leqq\beta<1)$ and for all $z \in U$, then we have $f(z) \in S(\alpha)$.

This is an improvement of Theorem A.

Taking $\beta=0$ in Corollary 1, we have

Corollary 2. If $f(z)\in A$ satisfies

$$\left|\frac{z^2 f''(z)}{f(z)}\right|<1-\alpha^2$$

for some $\alpha\ (0\leqq\alpha<1)$ and for all $z\in U$, then we have $f(z)\in S(\alpha)$.
This is an improvement of [1, Corollary 1].

Taking $\beta=1/2$ in Corollary 1, we have

Corollary 3. If $f(z)\in A$ satisfies

$$\left|\frac{z f'(z)}{f(z)}-1+\frac{z^2 f''(z)}{f(z)}\right|<(2-\alpha+\alpha^2)$$

for some $\alpha\ (0\leqq\alpha<1)$ and for all $z\in U$, then we have $f(z)\in S(\alpha)$.
This is an improvement of [1, Corollary 2].

Taking $\beta=0$ in Main theorem, we have

Corollary 4. Let $p(z)$ be analytic in $U$, $p(0)=1$ and suppose that

$$|p^2(z)-p(z)+z p'(z)|<1-\alpha^2$$

for all $z\in U$. Then we have

$$|p(z)-1|<1-\alpha$$

for all $z\in U$.

## References

[1] S. Fukui: A Remark on a Class of Certain Analytic Functions. Proc. Japan Acad., 66, Ser. A, 191-192(1990)

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[3] S. S. Miller and P. T. Mocanu: Second order differential inequalities in complex plane. J. Math. Anal. Appl., 65, 289-305 (1978)